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THE INFLUENCE OF PERTURBATIONS ON STABILITY IN TERMS OF TWO METRICS†

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The problem of stability under perturbations in terms of two metrics is investigated. Problems of the stability of the equilibrium position of a mechanical system with variable mass under perturbations are solved. © 1999 Elsevier Science Ltd. All rights reserved.

1. Consider a system described by the equations

$$\dot{y} = Y(t,y) + F(t,y), \quad Y, F \in C(R^+ \times E \to R^n) \tag{1.1}$$

where $R^+ = [0, +\infty[$, R^n is the n-dimensional space of y-vectors with norm $\|y\|$ and $E \subset R^n$ is an open domain. The function F expresses the action of certain perturbations, so that when there are no such perturbations the motion is described by the equations

$$\dot{y} = Y(t,y) \tag{1.2}$$

It is assumed that the function Y and F satisfy conditions guaranteeing the existence and uniqueness of solutions of systems (1.1) and (1.2), $y(t) = y(t, t_0, y_0)$ is a solution of system (1.1) and $\tilde{y}(t) = \tilde{y}(t, t_0, y_0)$ is a solution of system (1.2) such that $y(t, t_0, y_0) = y_0$, $\tilde{y}(t, t_0, y_0) = y_0$.

Let K be the class of functions of Hahn type, and let $h_0 \in C(R^n \times E \to R^+)$ and $h \in C^1(R^+ \times E \to R^+)$ be functions satisfying the following conditions

1. $\inf(h_0(t,y), t \in R^+ : t = \text{const}, y \in E) = 0$, $h(t,y) \not\equiv 0$;

2. $\exists\lambda > 0$, $\exists m \in K$, such that if $h_0(t,y) < \lambda$, then $h \leqslant m(h_0) \leqslant m(\lambda)$ (throughout, K is the class of functions of Hahn type [1]).

By introducing the functions h and h_0, the problem of stability in terms of two metrics [2] may be formulated in the following convenient manner [3].

Definition 1.1. System (1.2) is said to be (h_0, h)-stable if $\forall\epsilon > 0 \times (\forall t_0 \geqslant 0)\ (\exists\delta > 0)\ (\forall y_0;\ h_0(t_0, y_0) > \delta),\ h(t, \tilde{y}(t)) < \varepsilon \forall t \geqslant t_0)$.

Following [3, 4], we introduce the following definitions, corresponding to the definition of stability of the trivial solution under constantly acting perturbations [5], setting $S_q = \{(t,y) \in R^+ \times E : h(t,y) < q\}$, where $q = m(\lambda)$.

Definition 1.2. System (1.2) is said to be (h_0, h)-stable under constantly acting perturbations (CAP) if $(\forall\varepsilon > 0)(\forall t_0 \geqslant 0)(\exists\delta > 0)(\exists d > 0) \times (\forall y_0 : h_0(t_0, y_0) < \delta)\ (\forall F : \|F\| < d \text{ on } S_\varepsilon),\ (h(t, y(t)) < \varepsilon \forall t \geqslant t_0)$.

Definition 1.3. System (1.2) is said to be strongly (h_0, h)-stable under CAP if it is stable in the sense of Definition 1.2 and also $(\forall\varepsilon > 0)((\forall t_0 \geqslant 0)(\exists\delta > 0) \times (\forall\eta \in]0, \varepsilon[)(\exists d_1 \in]0, d[)(\forall y_0 : h_0(t_0, y_0) < \delta)$ $(\forall F : \|F\| < d_1 \text{ on } S_\varepsilon)\ (\exists T > 0),\ (h(t, y(t)) < \eta \forall t \geqslant t_0 + T)$.

If the numbers δ, d, d_1 and T in Definitions 1.1–1.3 are independent of t_0, we have the respective definitions of uniform (h_0, h)-stability under CAP.

Definition 1.4. System (1.2) is said to be (h_0, h)-stable under CAP small on the average if

$$(\forall\varepsilon > 0)(\forall t_0 \geqslant 0)(\forall T > 0)(\exists\delta > 0 : m(\delta) < \varepsilon)(\exists d > 0)(\forall y_0 : h(t_0, y_0) < \delta) \times$$

$$\times \left(\forall F : \int_{t=t_0}^{t_0+T} \sup(\| F(u,y) \| \text{ on } S_\varepsilon \right) du < d, \quad (h(t, \ y(t)) < \varepsilon \forall t \geqslant t_0).$$

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Definition 1.5. System (1.2) is said to be (h_0, h)-stable under integrally small CAP if the condition relating to F in Definition 1.1 is defined as

$$\left(\forall F: \int_{t=t_0}^{t_0+T} \sup(\| F(u,y) \| \text{ on } S_\varepsilon\right) du < d$$

2. We will now apply the method of Lyapunov functions to the problem as formulated.

Theorem 2.1. Assume that $h \in C^1(R^+ \times E \to R^+)$ and that a Lyapunov function $W = W(t, y, h) \in C^1(R^+ \times E \times R^+ \to R)$ exists satisfying the following conditions

1. $a(h) \leqslant W(t, y, h) \leqslant \alpha(t)b(h_0)$, $\dot{W}_{(1.2)}(t, y, h) \leqslant -\beta W(t, y, h)$, $\forall(t, y) \in S_q$, where $a, b \in K$, $\alpha(t) > 0$, $\beta = \text{const} = 0$;
2. a number $M > 0$ exists such that

$$\left\|\frac{\partial W}{\partial y}\right\|, \left|\frac{\partial W}{\partial h}\right|\left\|\frac{\partial h}{\partial y}\right\| \leqslant M \quad \forall(t, y) \in S_q.$$

Then system (1.1) is (h_0, h)-stable under CAP.

Proof. Given $\varepsilon > 0$ and $t_0 \geqslant 0$, define a number $\delta > 0$ such that $\alpha(t_0)b(\delta) = a(\varepsilon)$. For every $y_0 \in \{h_0(t_0, y) < \delta\}$ we then find $W(t_0, y_0, h(t_0, y_0)) < \alpha(t_0)b(\delta) < a(\varepsilon)$.

Let $y(t) = y(t, t_0, y_0)$, $y_0 \in \{h_0(t_0, y) < \delta\}$ be the solution of system (1.1) and let $W(t) = W(t, y(t), h(t, y(t))$ be a function defined on this equality.

We have $W(t_0) < a(\varepsilon)$. We will show that if the perturbation $F(t, y)$ satisfies the estimate $\|F(t, y)\| < \alpha a(\varepsilon)/(2M)$, then for all $t \geqslant t_0$ one has $h(t, y(t)) < \varepsilon$. For every such perturbation, we deduce from an estimate of the same type as the well-known Malkin relation [5] that

$$\dot{W}_{(1.1)} = \dot{W}_{(1.2)} + \sum_{i=1}^{n} \frac{\partial W}{\partial y_i} F_i + \frac{\partial W}{\partial h} \sum_{i=1}^{n} \frac{\partial h}{\partial y_i} F_i \leqslant -\alpha W + 2M \| F_i \| \leqslant -\alpha W + \alpha a(\varepsilon)$$

Thus, the function $W(t)$ satisfies the differential inequality

$$\dot{W}(t) \leqslant -\alpha W(t) + \alpha a(\varepsilon)$$

from which it follows that $a(h(t, y)(t, t_0, y_0)) \leqslant W(t) < a(\varepsilon)$ and accordingly that $h(t, y(t, t_0, x_0)) \leqslant \varepsilon$ for all $t \geqslant t_0$. The theorem is proved.

Theorem 2.2. Assume that instead of Condition 1 of Theorem 2.1, the following condition holds

$$a(h) \leqslant W(t, y, h) \leqslant b(h),\ \dot{W}(t, y, h) \leqslant -c(h) \forall(t, y) \in S_q, \text{ where the functions } a, b, c \in K.$$

Then system (1.1) is strongly uniformly (h_0, h)-stable under CAP.

The proof is analogous to that of Theorem 2.1.

3. Let us consider the above problem on the assumption that the right-hand sides of the unperturbed system (1.2) is bounded and satisfies a Lipschitz condition on every compact set $K \subset E$. Under these conditions, system (1.2) is precompact [6], so that the positive limit set of its solutions is quasi-invariant [6]. Using the technique of investigating stability properties on the basis of limit systems and Lyapunov functions with sign-definite derivative, proposed in [7, 8], one can also obtain definite results in the problem considered here.

Theorem 3.1. Assume that:

1. a domain $\Gamma_0 \subset E$, $\sup (h_0(t, y), t \geqslant 0, y \in \Gamma_0) \geqslant \lambda > 0$ exists such that the solutions of the perturbed system (1.1) in this domain are uniformly bounded by the finite domain Γ;
2. the solutions of the unperturbed system (1.2) in Γ are uniformly bounded by the compact domain Γ_1, $\Gamma_0 \subseteq \Gamma \subseteq \Gamma_1 \subset E$;
3. a Lyapunov function $W = W(t, y, h) \in C^1(R^+ \times E \times R^+ \to R)$ exists such that

$$a(h) \leqslant W(t,y,h) \leqslant b(h), \quad \dot{W}_{(1.2)}(t,y,h) \leqslant -V(t,y) \leqslant 0$$

$$\forall (t,y) \in S_q, \quad q = m(\lambda)$$

4. for each limit pair (Y, V) to (Φ, Ω) which is maximally invariant relative to the system $\dot{y} = \Phi(t, y)$, a subset of the set $\{\Omega(t, y) = 0\}$ is contained in the set $\{h(t, y) = 0\}$.

Then the perturbed system (1.1) is strongly uniformly (h_0, h)-stable under CAP.

Proof. Let us determine the properties of the unperturbed system (1.2). It follows from Condition 3 of the theorem, first of all, that the set $\{h(t, y) = 0\}$ is invariant and thus $Y(t, y) \equiv 0$ for $(t, y) \in \{h(t, y) = 0\}$.

Using Conditions 2–4 and following the proof of Theorem 2.4 in [8], we can now prove that the unperturbed system (1.2) is uniformly asymptotically (h_0, h)-stable and the domain Γ lies in the domain of uniform h-attraction.

We can now deduce from these properties of system (1.2), proceeding as in the proof of Theorem 2.1 of [3] or the Inversion Theorem 14.1 of [9], that in the domain Γ a function $W(t, y)$ exists satisfying the conditions of Theorem 2.2. Hence, by Condition 1 of the theorem, the desired result follows.

Example. The equations of motion of a point mass of variable mass along the Ox axis [10, 11] may be expressed as

$$(r(t)\dot{x}) = -f(t,x,\dot{x}) - p(t)g(x) + F(t,x,\dot{x}) \tag{3.1}$$

where $r(t)$ is the mass of the point, x is its coordinate and the right-hand side of the equation represents the action of all possible forces: reactive, frictional, potential and unknown perturbations.

We reduce Eq. (3.1) to the system

$$\dot{x} = y, \quad \dot{y} = -\frac{\dot{r}(t)}{r(t)}y - \frac{f(t,x,y)}{r(t)} - \frac{p(t)}{r(t)}g(x) + F_1(t,x,y) \tag{3.2}$$

and investigate the stability of (3.1) or (3.2) in terms of the two metrics

$$h_0 = \sup(|x|, |y|), \quad h(t,x,y) = 2\int_0^x g\tau d\tau + \frac{r(t)}{p(t)}y^2$$

Let us assume that the quantities occurring in (2.3) satisfy the conditions:

1. $g(x)x \geqslant 0, g(0) = 0, \int_0^x g(\tau)d\tau \to +\infty$ as $x \to +\infty$;
2. $r(t) > 0, p(t) > 0, 0 < m \leqslant (r(t)/p(t)) \leqslant M, (\dot{r}(t)/r(t) + \dot{p}(t)/p(t)) \geqslant l > 0 \forall t \in R^+$;
3. $f(t, x, y)y \geqslant a(|y|) \forall (t, x, y) \in R^+ \times R^2$;
4. the motions of the perturbed system in the domain $\{|\dot{x}_0| < H, |x_0| < H > 0\}$ are uniformly bounded.

Setting the Lyapunov function equal to $W = h$, we find that its derivative (in the absence of F_1) satisfies the estimate $\dot{W} = -a(|y|) \leqslant 0$. Hence, by Theorem 3.1, it follows that under these conditions the motion of the point is strongly uniformly (h_0, h)-stable under the CAP $F_1(t, x, y)$.

4. Let us consider the problem of the influence of other types of CAP on (h_0, h)-stability.

Theorem 4.1. Under the assumptions of Theorem 2.2, system (1.2) is uniformly (h_0, h)-stable under CAP that are small on the average.

Theorem 4.2. If the condition imposed in Theorem 2.2 on $\dot{W}_{(1.2)}$ is replaced by the weaker condition $\dot{W}_{(1.2)}(t, y, h) \leqslant 0$, with the other assumptions retained, then system (1.2) is uniformly (h_0, h)-stable under integrally small CAP.

The theorems are derived from Theorem 2.2, proceeding as in the proof of Theorem 4 of [12].

Example. The equations of motion of a holonomic mechanical system with N variable masses $m_r(t)$ under the action of potential, gyroscopic, dissipative and perturbing forces may be written as follows [10, 11]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = P_i + \sum_{j=1}^{n} q_{ij}\dot{q}_j + \frac{\partial U}{\partial q_i} - \frac{\partial R}{\partial \dot{q}_i} + F_i \tag{4.1}$$

where $q_1, q_2, \ldots, q_n$ are generalized coordinates, $T = T_2 + T_1 + T_0$ is the kinetic energy, P_i are the reactive forces, $g_{ij} = -g_{ij}(t, q)$ are the coefficients of the gyroscopic forces, $\partial U/\partial q_i$ are the potential forces, $2R = \Sigma_{i,j=1}^{n} b_{ij}\dot{q}_i\dot{q}_j$ are the dissipative forces and F_i are the perturbing forces.

Let us assume that the kinetic energy of the system, for which $\partial T/\partial t = 0$, satisfies the relation $T_0 + U \leqslant 0$, $T_0 + U = 0$ for $q = 0$, the separation and attachment of particles to points of the system are such that

$$\sum_{r=1}^{N} \dot{m}_r(\mathbf{V}_r \cdot \mathbf{v}_r) \leqslant 0$$

where $\dot{m}_r$ is the variation of the masses of the points of the system, $\vec{\mathbf{V}}_r$ and $\vec{\mathbf{v}}_r$ are the relative and translational velocities of the separating and attaching points and $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$ denotes the scalar product.

Set $h_0 = \sup(\|\dot{q}\|, |U|)$, $\|\dot{q}\|^2 = \dot{q}_1^2 + \dot{q}_2^2 + \ldots + \dot{q}_n^2$, $h = T_2 - T_0 - U$. For the derivative $\dot{W} = h$ when there are no perturbations F_i we find the limit

$$W = -2R + \sum_{r=1}^{N} \dot{m}_r(\mathbf{V}_r \cdot \mathbf{v}_r) \leqslant 0$$

By Theorem 4.1, it follows that system (4.1) will be uniformly (h_0, h)-stable under integrally small CAP.

The problem of (h_0, h)-stability under perturbations may also be solved by a method based on the principle of comparison with a vector-valued Lyapunov function [9, 10, 13].

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